A THEOREM ON CYCLIC POLYTOPES

BY

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ABSTRACT

Let $C(v, d)$ represent a cyclic polytope with v vertices in d dimensions. A criterion is given for deciding whether a given subset of the vertices of $C(v, d)$ is the set of vertices of some face of $C(v, d)$. This enables us to determine, in a simple manner, the number of j-faces of $C(v, d)$ for each value of j $(1 \leqq j \leqq d - 1)$.

Cyclic polytopes, which were discovered early this century by Carathéodory [1, 2] and more recently rediscovered by Gale [3] and Motzkin [5] as examples of neighbourly polytopes, play an important role in the combinatorial theory of convex polytopes. The main reason for this is the conjecture that, for a given number of vertices v and dimension d, the cyclic polytope $C(v, d)$ has the maximum possible number of faces of each dimension $k(1 \leqq k \leqq d - 1)$. For a short history, as well as further information about cyclic polytopes, the reader should consult the recent book [4] by Branko Grünbaum.

The purpose of this note is to prove a theorem which generalises Gale's evenness condition [4, 4.7.2]. It characterises, in a very simple manner, those subsets of the vertices of $C(v, d)$ which belong to a face of any dimension. The whole combinatorial structure of $C(v, d)$ thus becomes apparent, and all the well-known properties of cyclic polytopes are easy corollaries. In particular the theorem enables us to determine in a simple manner the numbers $f_k(C(v, d))$ of k-faces of $C(v, d)$. These numbers were first determined by Motzkin [5] but no proofs were given. For a proof using the Dehn-Sommerville equations, see [4, 9.6].

For brevity, any totally ordered set V with cardinality v will be called a v-set. For example, any v distinct points on a directed line, or simple arc, is a v-set. Write $V = \{x_1, \cdots, x_v\}$ where $x_i < x_j$ if and only if $i < j$. Then a subset $X \subseteq V$ is called *contiguous* if, for some $1 < i \leqq j < v$,

$$X = \{x_i, x_{i+1}, \cdots, x_j\},\ x_{i-1} \notin X,\ x_{j+1} \notin X.$$

X will be called *even* or *odd* according to the parity of card $X = j - i + 1$. An *end-set* is a subset Y of V of the form

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$$Y = \{x_1, \cdots, x_i\},\ x_{i+1} \notin Y, \text{ or}$$

$$Y = \{x_j, \cdots, x_v\},\ x_{j-1} \notin Y.$$

Clearly any subset $W \subseteq V$ $(W \neq V)$ can be written uniquely in the form

$$W = Y_1 \cup X_1 \cup \cdots \cup X_t \cup Y_2$$

where $0 \leqq t \leqq [\frac{1}{2}v]$, the X_i are contiguous subsets of V and Y_1, Y_2 are end-sets of V or are empty. W is said to be of type (r, s) if card $W = r$ and exactly s of the contiguous subsets X_i are odd.

The set of vertices $V = \operatorname{vert} C(v, d)$ of a cyclic polytope is a v-set, for it consists of v distinct points lying on a dth order curve μ in E^d. (In [4, 4.7], Grünbaum uses the moment curve $(t, t, \cdots t^d)$, $-\infty < t < \infty$, and mentions dth order curves that have been used by other authors.) With the above terminology, Gale's evenness condition [4, 4.7.2] can be stated as follows: *A subset $W \subseteq V$ is the set of vertices of a facet* (a$(d-1)$-dimensional face) *of $C(v,d)$ if and only if W is of type* $(d, 0)$. This is the particular case $k = d - 1$ of our theorem:

THEOREM. *A subset W of $V = \operatorname{vert} C(v, d)$ is the set of vertices of a k-face of $C(v, d)$ $(0 \leqq k \leqq d-1)$ if and only if W is of type $(k+1, s)$ with $s \leqq d - k - 1$.*

Proof. We consider first the case $k = d - 1$. Given any subset $W \subseteq V$ with card $W = d$, then W is an affinely independent subset of E^d, and the affine hull aff W is a hyperplane H. Since μ is of degree d, $H \cap \mu = W$, and the points of W divide μ into $d + 1$ arcs lying alternately on each side of H. Clearly conv W is a facet of $C(v, d)$ if and only if H supports $C(v, d)$, that is, if and only if every two points of $V \setminus W$ are separated, on μ, by an even number of points of W. This, in turn, is equivalent to the condition that W is of type $(d, 0)$, that is, W contains no odd contiguous subsets. (This proof is essentially the same as that of [4, 4.7.2].) The theorem is therefore true for $k = d - 1$. Consider now the general case. Let $W \subseteq V$ with card $W = k + 1$ $(0 \leqq k \leqq d - 1)$ be a given subset. If W has at most $d - k - 1$ odd contiguous subsets, then it is clearly possible to find a subset T of μ with $T \cap V = \emptyset$ and card $T = d - k - 1$ such that the subset $W \cup T$ of the $(v + d - k - 1)$-set $V \cup T$ has only even contiguous subsets. Then the hyperplane $H = \operatorname{aff}(W \cup T)$ supports $C(v, d)$ by the argument given above, and as $W \subseteq H$, conv W is a face of $C(v, d)$. The condition is also necessary, for if conv W is a face of $C(v, d)$, then it is also a face of some facet conv W' $(W \subseteq W' \subseteq V)$ of $C(v, d)$. Since W' has no odd contiguous subsets, clearly W can have at most $d - k - 1$ odd contiguous subsets.

Finally we note that since every set of $k + 1$ distinct points of $(1 \leqq k \leqq d - 1)$ are affinely independent, every k-face of $C(v, d)$ has $k + 1$ vertices (it is a k-simplex). This completes the proof of the theorem.

COROLLARY 1. *$C(v,d)$ is $[\frac{1}{2}d]$ – neighbourly. (That is, every $[\frac{1}{2}d]$ points of $V = \text{vert}\, C(v,d)$ are the vertices of a face of $C(v,d)$.)*

Proof. If card $W = [\frac{1}{2}d]$, then W has at most $[\frac{1}{2}d]$ odd contiguous subsets. Since $[\frac{1}{2}d] \leqq d - [\frac{1}{2}d]$, the theorem implies that conv W is a face of $C(v,d)$.

COROLLARY 2. *The number $f_k(C(v,d))$ of k faces of $C(v,d)$ is given by the expressions*

$$(1)\qquad f_k(C(v,2n)) = \sum_{j=1}^{n} \frac{v}{v-j}\binom{v-j}{j}\binom{j}{k+1-j}, \qquad 0 \leqq k \leqq 2n-1,$$

$$(2)\qquad f_k(C(v,2n+1)) = \sum_{j=0}^{n} \frac{k+2}{j+1}\binom{v-j-1}{j}\binom{j+1}{k+1-j}, \qquad 0 \leqq k \leqq 2n,$$

with the usual convention that a binomial coefficient $\binom{p}{q}$ is zero if $p < q$ or $q < 0$.

Proof. This depends upon a simple combinatorial argument to determine the number of distinct subsets $W \subseteq V$ of type $(k+1,s)$ with $s \leqq d-k-1$. The odd and even dimensional cases are essentially different: we begin with the case $d = 2n$.

By a *v-circuit* we mean any set of cardinality v which is cyclically ordered. For example v points on an oriented simple closed curve is a v-circuit. The essential feature of a v-circuit V is that every point of V has a uniquely defined successor; the vth successor of each point is the point itself. Contiguous subsets of a v-circuit are defined in the obvious manner, and a subset $W \subseteq V$ is said to be of type (r,s) if card $W = r$ and W contains exactly s odd contiguous subsets.

Let V be a v-set and $W \subseteq V$ be a subset of type $(k+1,s)$ or $(k+1,s-1)$, where s is any integer satisfying $s \equiv k+1 \pmod 2$. Then V may be made into a v-circuit by specifying that x_{i+1} is the successor of x_i, suffixes reduced modulo v, and W becomes a subset $W_1 \subseteq V_1$ of type $(k+1,s)$. (If W is of type $(k+1,s-1)$ then the fact that s and $k+1$ are of the same parity implies that the union of the end-sets of W has odd cardinality. Hence W_1 has one more odd contiguous subset than W.) We write $p(v,k+1,s)$ for the total number of distinct subsets $W \subseteq V$ (or of subsets $W_1 \subseteq V_1$) with the above properties.

In order to determine the numerical value of $p(v,k+1,s)$ we proceed as follows. By definition, each point $x \in V$ and therefore each contiguous subset of V_1, has a unique successor. Let $W_2 \subseteq V_1$ be the subset of type $(k+s+1,0)$ formed by adjoining to W_1 the successor of each of its s odd contiguous subsets. The number of such subsets W_2 is $p(v,k+s+1,0)$, and each W_2 consists of j pairs of adjacent points of V_1, where $2j = k+s+1$. Since deletion of the second point in any s of these j pairs produces a set of type $(k+1,s)$, we see that $\binom{j}{s}$ distinct sets W_1 correspond to the same W_2 and so

$$p(v,k+1,s) = \binom{j}{s} p(v,2j,0). \tag{3}$$

We shall now determine $p(v,2j,0)$. If we delete one point of each of the j pairs in a set W_2, we obtain a subset W_3 of a $(v-j)$-circuit V_2 with card $W_3 = j$. The number of such sets W_3 is clearly $\binom{v-j}{j}$. For each set W_3 let r be the number of cyclic permutations of V_2 (that is, automorphisms of V_2 which preserve the cyclic ordering of the points) which leave W_3 invariant. Then r is also the number of cyclic permutations of V_1 which leave W_2 invariant. The cyclic permutations of V_2 applied to W_3 yield $(v-j)/r$ distinct subsets of V_2, and cyclic permutations of V_1 applied to the corresponding set W_2 yield v/r distinct subsets of V_1 (of type $(2j,0)$). We deduce that

$$p(v,2j,0) = \frac{v}{v-j}\binom{v-j}{j}. \tag{4}$$

From (3) and (4),

$$p(v,k+1,s) = \frac{v}{v-j}\binom{v-j}{j}\binom{j}{s} \tag{5}$$

where $2j = k+s+1$, and from the theorem,

$$f_k(C(v,2n)) = \sum_{\substack{s=0 \\ s\equiv k+1 \pmod 2}}^{d-k-1} p(v,k+1,s). \tag{6}$$

Substituting the value of $p(v,k+1,s)$ from (5) in (6), and changing the variable in the summation from s to j, we immediately obtain (1). (The terms corresponding to $1 \leqq j \leqq [\frac{1}{2}(k+1)]$ in the summation are identically zero.)

For the odd-dimensional case $d = 2n+1$, we need to determine, for each k, with $0 \leqq k \leqq 2n$, the number of distinct subsets of V of type $(k+1,s)$ with $s \leqq 2n-k$. To do this we transform the v-set V into a $(v+1)$-circuit V_1, by adjoining one point x which is the successor of x_v and the predecessor of x_1. For each subset W of V we define $W_1 \subseteq V_1$ as that subset into which W is transformed, together with the additional point x. If W is of type $(k+1,s)$ or $(k+1,s-1)$ with $k \equiv s \pmod 2$, then W_1 is of type $(k+2,s)$. By (6) the number of such sets W_1 with $s \leqq 2n-k$ is

$$(7) \qquad \sum_{\substack{s=0 \\ s\equiv k \pmod 2}}^{2n-k} p(v+1,k+2,s) = f_{k+1}(C(v+1,2n+2)).$$

For each W_1 of type $(k+2,s)$ let r be the number of cyclic permutations of V_1 that leave W_1 invariant. Then cyclic permutations of V_1 applied to W_1 yield $(v+1)/r$ distinct subsets of V_1 of type $(k+1,s)$. Since deletion of any one of the $k+2$ points of W_1 converts V_1 into a v-set V, we see that each W_1 leads to $(k+2)/r$ distinct subsets $W \subseteq V$ of type $(k+1,s)$ or $(k+1,s-1)$. Hence from (7), the total number of distinct subsets of V of type $(k+1,s)$ with $s \leqq 2n-k$ is

$$f_k(C(v,2n+1)) = \frac{k+2}{v+1} f_{k+1}(C(v+1,2n+2))$$

(compare [1, 9.6.2]). If we substitute for $f_{k+1}(C(v+1,2n+2))$ from (1) we obtain (2), which completes the proof of the corollary.

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References

1. C. Carathéodory, *Ueber den Variabilitätsbereich der Koeffizienten von Potentzreiher die gegebene Werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.

2. C. Carathéodory, *Ueber den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen* Rend. Circ. Mat. Palermo, **32** (1911), 193–217.

3. D. Gale, *Neighborly and cyclic polytopes*, Proc. Symp. Pure Math., **7** (Convexity) (1963), 225–232.

4. B. Grünbaum, *Convex Polytopes*, London-New York-Sydney, 1967.

5. T. S. Motzkin, *Comonotone curves and polyhedra*, Abstract 111, Bull. Am. Math. Soc., **63** (1957), 35.

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